A Sharp Condition for the Loewner Equation to Generate Slits

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Abstract

D. Marshall and S. Rohde have recently shown that there exists $C_0 > 0$ so that the Loewner equation generates slits whenever the driving term is Hölder continuous with exponent $\frac{1}{2}$ and norm less than C_0 [11]. In this paper, we show that the maximal value for C_0 is 4.

Introduction

When Loewner introduced his namesake differential equation in 1923, it greatly impacted the theory of univalent functions. A univalent function f is a conformal map of the unit disc, normalized by $f(0) = 0$ and $f'(0) = 1$. In other words, it has the following power series representation in the unit disc:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots .$$

In 1916 Bieberbach [2] had shown that $|a_2| \leq 2$ and had conjectured that $|a_n| \leq n$ for all n. It was Loewner's differential equation that led to a proof of the case $n = 3$ in 1923. See [1] or [5] for a proof of this and for more classical applications of the Loewner equation. When the Bieberbach conjecture finally was proved for general n in 1985 by de Branges [4], the Loewner equation again played a key role.

In addition to its importance in the theory of univalent functions, the Loewner differential equation has gained recent prominence with the introduction of a stochastic process called "Stochastic Loewner Evolution", or SLE, by O. Schramm [13]. Many results in this fast-growing field can be found in the recent work of mathematicians such as Lawler, Rohde, Schramm, Smirnov, and Werner. See [7] for a survey paper with an extensive bibliography.

In the next two sections, we will introduce two formulations of the deterministic Loewner differential equation, the halfplane version and the disc version. This is followed by a discussion of some problems associated with the geometry of the solutions to the Loewner equation. The rest of the paper is concerned with proving Theorem 2 below, which builds upon D. Marshall and S. Rohde's recent work [11] concerning when the Loewner equation can generate slits. The fifth section contains examples and lemmas related to a natural obstacle to generating slits, the sixth section includes lemmas about conformal welding and

the Loewner equation, and the final section is the proof of Theorem 3, which is equivalent to Theorem 2.

The Loewner equation in the halfplane

Let $\gamma(t)$ be a simple continuous curve in $\mathbb{H} \cup \{0\}$ with $\gamma(0) = 0$ and $t \in [0, T]$. Then there is a unique conformal map $g_t : \mathbb{H} \setminus \gamma[0, t] \to \mathbb{H}$ with the following normalization, called the hydrodynamic normalization, near infinity:

$$g_t(z) = z + \frac{c(t)}{z} + O\left(\frac{1}{z^2}\right).$$

It is an easy exercise to check that $c(t)$ is continuously increasing in t and that $c(0) = 0$. Therefore γ can be reparametrized so that $c(t) = 2t$. Assuming this normalization, one can show that g_t satisfies the following form of Loewner's differential equation: for all $t \in [0, T]$ and all $z \in \mathbb{H} \setminus \gamma[0, t]$,

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \lambda(t)},$$

$$g_0(z) = z,$$

where λ is a continuous, real-valued function. Further, it can be shown that g_t extends continuously to $\gamma(t)$ and $g_t(\gamma(t))$ equals $\lambda(t)$.

On the other hand, if we start with a continuous $\lambda : [0, T] \to \mathbb{R}$, we can consider the following initial value problem for each $z \in \mathbb{H}$:

$$\begin{aligned} \frac{\partial}{\partial t} g(t, z) &= \frac{2}{g(t, z) - \lambda(t)}, \\ g(0, z) &= z. \end{aligned} \tag{1}$$

For each $z \in \mathbb{H}$ there is some time interval $[0, s)$ for which a solution $g(t, z)$ exists. Let $T_z = \sup\{s \in [0, T] : g(t, z) \text{ exists on } [0, s)\}$. Set $G_t = \{z \in \mathbb{H} : T_z > t\}$ and $g_t(z) = g(t, z)$. Then one can prove that the set G_t is a simply connected subdomain of $\mathbb{H}$ and g_t is the unique conformal map from G_t onto $\mathbb{H}$ with the following normalization near infinity:

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right).$$

The function $\lambda(t)$ is called the driving term, and the domains G_t as well as the functions g_t are said to be generated by λ.

The domains G_t generated by a continuous driving term λ are not necessarily slit-halfplanes, i.e. domains of the form $\mathbb{H} \setminus \gamma[0, t]$, for some simple continuous curve γ in $\mathbb{H} \cup \{\gamma(0)\}$ with $\gamma(0) \in \mathbb{R}$. We will give an example later in the paper where a non-slit-halfplane is generated by a driving term which is not only continuous but also is in $\text{Lip}(\frac{1}{2})$. Recall that $\text{Lip}(\frac{1}{2})$ is the space of Hölder

continuous functions with exponent $\frac{1}{2}$, that is the space of functions $\lambda(t)$ satisfying $|\lambda(s) - \lambda(t)| \leq c|s-t|^{1/2}$, with $\|\lambda\|_{\frac{1}{2}}$ denoting the smallest such c. The necessary and sufficient condition for a decreasing family of domains $\{G_t\}$ to be generated by a continuous driving term can be found in Section 2.3 of [10].

The Loewner equation in the disc

The setup for the disc version of the Loewner equation is similar to that of the halfplane version, but the normalization will be at an interior point rather than at a boundary point. For the unit disc $\mathbb{D}$ slit by a simple curve $\gamma(t)$ in $\mathbb{D} \cup \{1\}$ with $\gamma(0) = 1$ and $\gamma(t) \neq 0$ for any t, there is a unique family of conformal maps $\{g_t\}$ so that $g_t : \mathbb{D} \setminus \gamma[0,t] \to \mathbb{D}$ with the normalizations $g_t(0) = 0$ and $g_t'(0) > 0$. Further, by reparametrizing γ if necessary, we can assume that $g_t'(0) = e^t$. If we again set $\lambda(t) = g_t(\gamma(t))$, then

$$\frac{\partial}{\partial t} g_t(z) = g_t(z) \frac{\lambda(t) + g_t(z)}{\lambda(t) - g_t(z)}, \qquad (2)$$
$$g_0(z) = z.$$

Given any continuous function $\lambda : [0,T] \to \partial\mathbb{D}$, we can solve the initial value problem (2) for $z \in \mathbb{D}$. As in the halfplane version, this will generate a family of conformal maps $\{g_t\}$ which map from a simply connected subdomain of the unit disc onto the unit disc and which are normalized by $g_t(0) = 0$ and $g_t'(0) = e^t$.

Some results

We return to the halfplane version of the Loewner equation, which will be the setting for the rest of this paper. For $\kappa \geq 0$, set $\lambda(t) = \sqrt{\kappa} B_t$, where B_t is standard Brownian motion. Then chordal SLE$_\kappa$ is the random family of conformal maps generated by λ, that is, the family of maps solving the following stochastic differential equation:

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t},$$
$$g_0(z) = z.$$

For SLE, it is possible to define an almost surely continuous path $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ such that the domains G_t generated by $\lambda(t) = \sqrt{\kappa} B_t$ are the unbounded components of $\mathbb{H} \setminus \gamma[0,t]$ for every $t \geq 0$. See [12] and, for the case $\kappa = 8$, [9]. Further, S. Rohde and O. Schramm [12] have shown the following classification:

1. For $\kappa \in [0,4]$, $\gamma(t)$ is almost surely a simple path contained in $\mathbb{H} \cup \{0\}$.
2. For $\kappa \in (4,8)$, $\gamma(t)$ is almost surely a non-simple path.
3. For $\kappa \in [8,\infty)$, $\gamma(t)$ is almost surely a space-filling curve.

This result motivates a question in the deterministic setting. Can we classify the kinds of domains generated by a driving term λ in terms of some characteristic of λ? There is only a partial understanding of this question. In the case of a domain slit by an analytic slit, the driving term is real analytic, and if the slit is C^n then the driving term is at least C^{n-1}. See [6] and [3].

D. Marshall and S. Rohde address the question of when the generated domains G_t are quasislit-halfplanes in [11], where a quasislit-halfplane is the image of $\mathbb{H} \setminus [0, i]$ under a quasiconformal mapping fixing $\mathbb{H}$ and ∞. They prove the following:

Theorem 1. *If G_t is a quasislit-halfplane for all t, then $\lambda \in Lip(\frac{1}{2})$. Conversely, there exists C_0 such that if the driving term $\lambda \in Lip(\frac{1}{2})$ with $\|\lambda\|_{\frac{1}{2}} < C_0$, then G_t is a quasislit-halfplane for all t.*

Although they work with the technically more challenging disc version of the Loewner equation, their techniques carry over to prove the result in the halfplane version as well. In the remainder of this paper, working with the halfplane version of Loewner's equation, we will show that the maximal value for C_0 is 4.

Theorem 2. *If $\lambda \in Lip(\frac{1}{2})$ with $\|\lambda\|_{\frac{1}{2}} < 4$, then the domains G_t generated by λ are quasislit-halfplanes.*

Further, for each $c \geq 4$, there exists a driving term $\lambda \in \mathrm{Lip}(\frac{1}{2})$ with $\|\lambda\|_{\frac{1}{2}} = c$ so that λ does not generate slit-halfplanes. We will see examples of this in the next section. Similar examples were discovered independently by L. Kadanoff, W. Kager, and B. Nienhuis [8]. Their work also includes descriptions and pictures of the generated domains.

There is another version of the Loewner equation in the halfplane. Let $\xi : [0, T] \to \mathbb{R}$ be continuous and consider the following initial value problem, in which a negative sign has been introduced on the righthand side of (1):

$$\frac{\partial}{\partial t} f(t, z) = \frac{-2}{f(t, z) - \xi(t)}, \tag{3}$$
$$f(0, z) = z$$

for $z \in \mathbb{H}$. In this case, for each $z \in \mathbb{H}$, the solution $f(t, z)$ exists for all $t \in [0, T]$. Setting $f_t(z) = f(t, z)$, we have that f_t is defined on all of $\mathbb{H}$. As in the previous case, it can be shown that f_t is a conformal map from $\mathbb{H}$ into $\mathbb{H}$, and near infinity it has the form

$$f_t(z) = z + \frac{-2t}{z} + O(\frac{1}{z^2}).$$

We think of the funtions f_t as being generated by "running time backwards."

These two forms of Loewner's differential equation are related. Given a continuous function λ on $[0, T]$, set $\xi(t) = \lambda(T - t)$. Let g_t be the functions generated by λ from (1), and let f_t be the functions generated by ξ from (3). It is

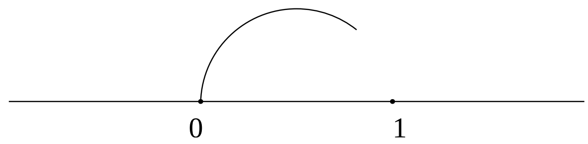

Figure 1: One of the domains generated by $\lambda(t) = \frac{3}{2} - \frac{3}{2}\sqrt{1-8t}$

not true that $f_t(z) = g_t^{-1}(z)$ for all $t \in [0, T]$, but it is true that $f_T(z) = g_T^{-1}(z)$. Therefore Theorem 2 is equivalent to the following:

Theorem 3. *If $\xi \in Lip(\frac{1}{2})$ with $\|\xi\|_{\frac{1}{2}} < 4$, then $f_t(\mathbb{H})$ is a quasislit-halfplane for all t, where f_t are the maps generated by ξ.*

When the singularity catches solutions

Let $\lambda \in \mathrm{Lip}(\frac{1}{2})$ and suppose that the domains G_t generated by λ are slit-halfplanes. Then the maps g_t extend continuously to $\mathbb{R} \setminus \{\lambda(0)\}$. Thus for each $x_0 \in \mathbb{R} \setminus \{\lambda(0)\}$, $x(t) := g_t(x_0)$ is a solution to the following real-valued initial value problem:

$$\frac{\partial}{\partial t} x(t) = \frac{2}{x(t) - \lambda(t)}, \qquad x(0) = x_0. \tag{4}$$

Further, if λ is defined on $[0, T]$, then $x(t) \neq \lambda(t)$ for any $t \in [0, T]$, since otherwise (4) would fail to have a solution for all $t \in [0, T]$.

Note that if $x_0 > \lambda(0)$, then $\frac{\partial}{\partial t} x(t) > 0$ as long as $x(t) \neq \lambda(t)$. So two things can happen: either $x(t)$ continues to move to the right, staying strictly larger than the driving term, or the driving term moves fast enough to "catch" $x(t)$ and there is some time t_0 where $x(t_0) = \lambda(t_0)$. The case when $x_0 < \lambda(0)$ is similar but with $x(t)$ moving to the left. Thus, when the domains generated are slit-halfplanes, we see that $\lambda(t)$ cannot "catch" any solution $x(t)$ to (4).

To build our intuition, let us briefly consider a particular example. Let $G_t = \mathbb{H} \setminus \gamma[0, t]$, where γ parametrizes the upper half-circle of radius $\frac{1}{2}$ centered at $\frac{1}{2}$, as pictured in Figure 1. In this case it is possible, although unpleasant, to compute the maps g_t and to ascertain that the driving term generating this scenario is the function $\lambda(t) = \frac{3}{2} - \frac{3}{2}\sqrt{1-8t}$, for $t \in [0, \frac{1}{8}]$. The time $t = \frac{1}{8}$ corresponds to the moment that the circular arc touches back on the real line, and $G_{\frac{1}{8}} = \mathbb{H} \setminus D(\frac{1}{2}, \frac{1}{2})$.

For $t \in [0, \frac{1}{8} - \epsilon]$, the domains G_t are slit-halfplanes, and therefore for any $x_0 \neq 0$, the solutions $x(t)$ to (4) exist on this time interval. What happens to these solutions when $t = \frac{1}{8}$? Clearly, $g_{\frac{1}{8}}$ extends only to $\mathbb{R} \setminus [0, 1]$. That is, on $[0, \frac{1}{8}]$, solutions to (4) exist only for $x_0 > 1$ or $x_0 < 0$. So if $x_0 \in (0, 1]$, the function $x(t)$ resulting from (4) must be "caught" by λ at time $t = \frac{1}{8}$. For

example, it is easy to check that the solution to (4) when $x_0 = 1$ is $x(t) = \frac{3}{2} - \frac{1}{2}\sqrt{1-8t}$. Here we see that $x(\frac{1}{8}) = \frac{3}{2} = \lambda(\frac{1}{8})$.

To determine an upper bound on the constant C_0 in Theorem 1, we can analyze the situations where this "catching" could occur, since this implies that the family of domains G_t is not a family of slit-halfplanes. In the example above, $\|\lambda\|_{\frac{1}{2}} = 3\sqrt{2}$, which indicates that $C_0 \leq 3\sqrt{2}$. Moreover, for any $c \geq 4$, it is easy to give an example of a driving term λ with $\|\lambda\|_{\frac{1}{2}} = c$ so that λ can "catch" a function $x(t)$ generated by (4) for some x_0. Let $\lambda(t) = c - c\sqrt{1-t}$ and $x(t) = c - a\sqrt{1-t}$ where $a = \frac{1}{2}(c + \sqrt{c^2-16})$. In particular, when $c = 4$, then $\lambda(t) = 4 - 4\sqrt{1-t}$ and $x(t) = 4 - 2\sqrt{1-t}$. One can check that $x(t)$ is a solution to (4) with $x_0 = c - a > 0$. However $x(1) = c = \lambda(1)$. Therefore, since $\lambda(t)$ has "caught" $x(t)$, λ cannot generate slit-halfplanes. This implies that the constant C_0 in Theorem 1 cannot be greater than 4.

In contrast to the examples above, the following lemma shows that if λ can "catch" some $x(t)$, then $\|\lambda\|_{\frac{1}{2}} \geq 4$. To make things slightly simplier, we take advantage of the fact that the halfplane version of the Loewner equation satisfies a useful scaling property: If $\lambda(t)$ and $x(t)$ satisfy equation (4), then $\hat{\lambda}(t) := \frac{1}{r}\lambda(r^2 t)$ and $\hat{x}(t) := \frac{1}{r}x(r^2 t)$ also satisfy equation (4). Verifying this is an easy exercise. Using this scaling property, we can assume that if a "catching" occurs, then it happens at time 1. More precisely, if $x(t_0) = \lambda(t_0)$ and $x(t) \neq \lambda(t)$ for $t < t_0$, then without loss of generality $t_0 = 1$. Also, nothing is lost by assuming that $\lambda(0) = 0$ and $x_0 > 0$.

Lemma 1. *Let $\lambda \in Lip(\frac{1}{2})$ with $\lambda(0) = 0$ and let $x_0 > 0$. Suppose that $x(t)$ is a solution to* (4) *and that $x(1) = \lambda(1)$. Then $\|\lambda\|_{\frac{1}{2}} \geq 4$.*

Proof. Let c=$\|\lambda\|_{\frac{1}{2}}$. From (4), we have that $x(t)$ is increasing in t. So then since $\lambda \in \text{Lip}(\frac{1}{2})$,

$$x(t) - \lambda(t) \leq x(1) - \lambda(1) + c\sqrt{1-t} \leq c\sqrt{1-t}.$$

From (4) we have

$$\dot{x}(t) \geq \frac{2}{c\sqrt{1-t}}.$$

Integrating gives that

$$x(1) - x(t) \geq \frac{4}{c}\sqrt{1-t}.$$

Letting $t = 0$ and using that $x(1) - x_0 < c$, we see that $c - \frac{4}{c} > 0$ and so $c > 2$. But we also have a better estimate for $x(t)$:

$$x(t) \leq x(1) - \frac{4}{c}\sqrt{1-t}.$$

Now using this estimate, we can repeat the above argument. So

$$x(t) - \lambda(t) \leq (c - \frac{4}{c})\sqrt{1-t},$$

which leads to a new estimate for $\dot{x}(t)$. Then by integration,

$$x(1) - x(t) \geq \frac{4}{c - \frac{4}{c}} \sqrt{1-t}.$$

This implies that $c - \frac{4}{c - \frac{4}{c}} > 0$ and so $c > 2\sqrt{2}$. Again we also get an improved estimate for $x(t)$:

$$x(t) \leq x(1) - \frac{4}{c - \frac{4}{c}} \sqrt{1-t}.$$

Repeating this procedure n times gives that $h_n(c) > 0$ where h_n is recursively defined as follows:

$$h_1(x) = x - \frac{4}{x},$$

$$h_n(x) = x - \frac{4}{h_{n-1}(x)}.$$

Note that $h_1(x)$ is an increasing function from $(0, \infty)$ onto $\mathbb{R}$. It is easy to show inductively that we can define an increasing sequence $\{x_n\}$ so that $h_n(x_n) = 0$, and $h_{n+1}(x)$ is an increasing function from (x_n, ∞) onto $\mathbb{R}$. Note that we have shown that $x_1 = 2$ and $x_2 = 2\sqrt{2}$. Since $h_n(c) > 0$ for all n, $c > x_n$ for all n. It simply remains to show that $x_n \nearrow 4$.

An easy inductive argument gives that $h_n(4) \geq 2$ for all n. If $4 \in (x_{k-1}, x_k]$ for some k, then $h_k(4) \leq 0$. Therefore, the increasing sequence $\{x_n\}$ is bounded above by 4, and hence there exists some $a \leq 4$ such that $x_n \nearrow a$. Now, $h_n(a) > h_n(x_n) = 0$ for all n. If $h_k(a) \leq 1$ for some k, then $h_{k+1}(a) = a - \frac{4}{h_n(a)} \leq 0$. So we must have $h_n(a) > 1$ for all n. Since $h_n(a)$ is decreasing in n and bounded below by 1, $h_n(a) \searrow L$ for some $L \geq 1$. So then,

$$L = \lim_{n\to\infty} h_n(a) = \lim_{n\to\infty} a - \frac{4}{h_{n-1}(a)} = a - \frac{4}{L}.$$

Solving the above for L gives that

$$L = \frac{a \pm \sqrt{a^2 - 16}}{2}.$$

Since we know the real-valued limit L exists, we must have $a \geq 4$. Hence, $a = 4$, completing the proof.

$\square$

Note that in the proof above, we have also shown the following: if $h_n(c) > 0$ for all n, then $c \geq 4$. This follows since $h_n(c) > 0$ for all n implies that $c > x_n$ for all n and since $x_n \nearrow 4$. We mention this here, since we will use this fact in the proof of the next lemma.

Although Lemma 1 certainly suggests that the maximal value for C_0 is 4, it is not a proof of Theorem 2. In theory, there may be more obstacles to generating quasislit-halfplanes than that of the driving term catching up to some solution

to (4). However, we will see that this is basically the only obstacle. Refining the above argument gives Lemma 2, which combined with techniques in [11] will lead to the proof of Theorem 2. The idea of Lemma 2 is that if λ can get close to catching some $x(t)$, then $\|\lambda\|_{\frac{1}{2}}$ must be close to being greater than or equal to 4.

Lemma 2. *Let $\lambda \in Lip(\frac{1}{2})$ with $\lambda(0) = 0$ and $\|\lambda\|_{\frac{1}{2}} < 4$. Then there exists $\epsilon = \epsilon(\|\lambda\|_{\frac{1}{2}}) > 0$ so that $x(1) - \lambda(1) > \epsilon$, where $x(t)$ is the solution to (4) with $x_0 > 0$.*

Proof. Suppose $x(t)$ is a solution to (4) for some $x_0 > 0$ so that $x(1) - \lambda(1) \leq \epsilon$. We will show that there exists some $\epsilon > 0$ so that this leads to a contradiction. Again, let $c = \|\lambda\|_{\frac{1}{2}}$. As in the previous proof, define h_n recursively by

$$h_1(c) = c - \frac{4}{c},$$

$$h_n(c) = c - \frac{4}{h_{n-1}(c)}.$$

Since $c < 4$, there is some minimal n so that $h_n(c) \leq 0$ (see the comment following the proof of Lemma 1.) If $h_n(c) = 0$, replace c with a slightly larger value, that is, recalling our notation from the previous proof, replace c with some number in the interval (x_n, x_{n+1}). Then $h_{n+1}(c) < 0$. We stop once we are in the case that $h_k(c) < 0$.

Also recursivly define e_n by

$$e_1(c, \epsilon) = \epsilon + \frac{4\epsilon}{c^2} \ln(1 + \frac{c}{\epsilon}),$$

$$e_n(c, \epsilon) = \epsilon + \frac{4e_{n-1}(c, \epsilon)}{(h_{n-1}(c))^2} \ln(1 + \frac{h_{n-1}(c)}{e_{n-1}(c, \epsilon)}).$$

The recursive definition for e_n is unpleasant, but all that we shall need is that for c and n fixed, $e_n(c, \epsilon) \to 0$ as $\epsilon \to 0$. This is easy to verify by induction.

To begin, we will prove by induction that

$$x(1) - x(t) \geq \epsilon - e_n(c, \epsilon) + (c - h_n(c))\sqrt{1 - t}. \tag{5}$$

First we show equation (5) when $n = 1$. We have

$$x(t) - \lambda(t) \leq x(1) - \lambda(1) + c\sqrt{1 - t} \leq \epsilon + c\sqrt{1 - t}$$

which implies that

$$\dot{x}(t) \geq \frac{2}{\epsilon + c\sqrt{1 - t}}.$$

Since

$$\int_t^1 \frac{2}{a + b\sqrt{1 - s}} ds = \frac{4}{b}\sqrt{1 - t} - \frac{4a}{b^2} \ln(1 + \frac{b}{a}\sqrt{1 - t}),$$

integrating gives

$$x(1) - x(t) \geq \frac{4}{c}\sqrt{1-t} - \frac{4\epsilon}{c^2}\ln(1 + \frac{c}{\epsilon}\sqrt{1-t}),$$

and so, as desired (5) holds for $n = 1$.

Next assume equation (5) holds for $n = k$. Then

$$x(t) \leq x(1) - \epsilon + e_k(c,\epsilon) + (h_k(c) - c)\sqrt{1-t},$$

and so

$$x(t) - \lambda(t) \leq e_k(c,\epsilon) + h_k(c)\sqrt{1-t}.$$

This again gives us an esimate for $\dot{x}(t)$ and integrating yields

$$x(1) - x(t) \geq \frac{4}{h_k(c)}\sqrt{1-t} - \frac{4e_k(c,\epsilon)}{h_k(c)^2}\ln(1 + \frac{h_k(c)}{e_k(c,\epsilon)}\sqrt{1-t}).$$

Thus equation (5) holds for $n = k+1$, completing our verification of (5) by induction.

Recall that $x(1) \leq c + \epsilon$. Thus letting $t = 0$ in equation (5) gives

$$h_n(c) + e_n(c,\epsilon) > 0.$$

As mentioned before, by adjusting c slightly if necessary, there is some n such that $h_n(c) < 0$. Then since $e_n(c,\epsilon) \to 0$ as $\epsilon \to 0$, there exists some $\epsilon > 0$ so that $e_n(c,\epsilon) < -h_n(c)$. But this contradicts the fact that $h_n(c) + e_n(c,\epsilon) > 0$. Therefore, there exists $\epsilon > 0$ so that $x(1) - \lambda(1) > \epsilon$, for $x(t)$ the solution to (4) with $x_0 > 0$.

$\square$

Now we wish to run time backwards, and so we must consider the second form of the Loewner equation in the upper halfplane. Recall that from (3), the driving term $\xi(t)$ generates conformal functions f_t, which map from $\mathbb{H}$ into $\mathbb{H}$. If the image of f_t is a quasislit-halfplane, then we can extend f_t continuously to $\mathbb{R}$, and for each $x_0 \in \mathbb{R} \setminus \{\xi(0)\}$, $x(t) := f_t(x)$ is a solution to

$$\frac{\partial}{\partial t}x(t) = \frac{-2}{x(t) - \xi(t)}, \tag{6}$$
$$x(0) = x_0.$$

Note that the solution $x(t)$ might not exist for all time. Indeed, in the case that $\|\xi\|_{\frac{1}{2}} < 4$, the following corollary shows that $x(t)$ will hit the singularity $\xi(t)$ in finite time. We define the hitting time $T(x_0)$ to be the first time that $x(t)$ equals $\xi(t)$, that is, $x(T(x_0)) = \xi(T(x_0))$ and $x(t) \neq \xi(t)$ for $t < T(x_0)$. If $x(t)$ never equals $\xi(t)$, then $T(x_0) := \infty$.

Corollary 1. *Let $\xi \in Lip(\frac{1}{2})$ with $\|\xi\|_{\frac{1}{2}} < 4$ and $\xi(0) = 0$. Suppose that $x(t)$ is a solution to (6), with $x_0 \neq 0$. Then $K_1 x_0^2 \leq T(x_0) \leq K_2 x_0^2$, where $0 < K_i = K_i(\|\xi\|_{\frac{1}{2}}) < \infty$.*

Proof. For $c = \|\xi\|_{\frac{1}{2}}$, let $\epsilon = \epsilon_c > 0$ be given as in Lemma 2, and let $x(t)$ be the solution to (6) with $x(0) = \epsilon$. If $T(\epsilon) > 1$, then $\lambda(t) = \xi(1-t) - \xi(1)$ and $y(t) = x(1-t) - \xi(1)$ satisfy the differential equation (4), with $y(0) = x(1) - \xi(1) > 0$. Thus Lemma 2 implies that $\epsilon = y(1) - \lambda(1) > \epsilon$. This is a contradiction, and so $T(\epsilon) \leq 1$.

Now suppose $x_0 > 0$, with $x(t)$ again the corresponding solution to (6). Then by the scaling property, $\hat{\xi}(t)$ and $\hat{x}(t)$ satisfy equation (6), where

$$\hat{\xi}(t) := \frac{\epsilon}{x_0}\xi(\frac{x_0^2}{\epsilon^2}t),$$

and

$$\hat{x}(t) := \frac{\epsilon}{x_0}x(\frac{x_0^2}{\epsilon^2}t).$$

Note that $\hat{x}(0) = \epsilon$. Therefore $T(x_0) = \frac{x_0^2}{\epsilon^2}T(\epsilon) \leq K_2 x_0^2$ where $K_2 = K_2(c) < \infty$.

For the lower bound, assume first that $x_0 = 1$, and assume that $T(1) = \delta$ is small. Then since $\xi(t) \leq c\sqrt{t}$, we have $x(\delta) \leq c\sqrt{\delta}$. Taking δ small enough so that $c\sqrt{\delta} < \frac{1}{2}$, let t_0 be the time when $x(t) = \frac{1}{2}$. Then,

$$-\frac{1}{2} = \int_0^{t_0} \frac{-2}{x(s) - \xi(s)}ds \geq \frac{-2t_0}{\frac{1}{2} - c\sqrt{\delta}},$$

and so,

$$\frac{1}{2}(\frac{1}{2} - c\sqrt{\delta}) \leq 2\delta.$$

This leads to a contradiction if δ is sufficiently small. Therefore $T(1) \geq K_1$ for some $K_1 = K_1(c) > 0$. Then by the scaling property, $T(x_0) \geq K_1 x_0^2$.

□

In the previous corollary, we saw that if $\|\xi\|_{\frac{1}{2}} < 4$ then solutions $x(t)$ to (6) will hit the singularity in finite time. Lemma 3 shows that there is more that is true. For each finite time, there are exactly two initial points, one on each side of the singularity, so that the solutions to (6) will hit the singularity at that time.

Lemma 3. *Let $\xi \in Lip(\frac{1}{2})$ with $\|\xi\|_{\frac{1}{2}} < 4$. For each $T > 0$, there exist exactly two real numbers $x_0, \hat{x}_0$ so that $x(T) = \hat{x}(T) = \xi(T)$.*

Proof. First notice that no two points on the same side of the singularity can give rise to solutions to (6) that will hit at the same time. This follows from the fact that $\delta(t) := y(t) - x(t)$ is increasing in t for $\xi(0) < x_0 < y_0$, since

$$\dot{\delta}(t) = 2\frac{y(t) - x(t)}{(y(t) - \xi(t))(x(t) - \xi(t))}.$$

Thus there are at most two points that can hit at time T.

Next we'll show that there is one point x_0 to the right of the singularity with $x(T) = \xi(T)$. For each $n \in \mathbb{N}$, set $w_n = \xi(T) + \frac{1}{n}$. Now, starting at w_n, run time from T back to 0. This corresponds to solving (4) with intial value w_n. Since $\|\xi\|_{\frac{1}{2}} < 4$, the driving term cannot catch up with this solution, $g_t(w_n)$, by Lemma 1, and so it is well-defined up through time T. Thus, $x_n := g_T(w_n) = f_T^{-1}(w_n)$ is well-defined. Further, by Lemma 2, there exists $\epsilon > 0$ so that $x_n - \xi(0) > \epsilon$. Therefore, $\{x_n\}$ is a decreasing sequence bounded below by $\xi(0) + \epsilon$, and so it has a limit x_0. Then $x_0 > \xi(0)$ and clearly we have $x(T) = \xi(T)$. This completes the proof.

□

Conformal welding with the Loewner equation

The previous lemma allows us to define the welding homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ as the orientation-reversing map that satisfies $\phi(x) = y$ if and only if $T(x) = T(y)$. Thus the welding homeomorphism interchanges the two points which hit the singularity at the same time. Note that if ξ is not defined for all time, but for a finite interval $[0, T]$, the welding homeomorphism will not be defined on all $\mathbb{R}$. However, we can overcome this technicality by setting $\xi(t) := \xi(T)$ for $t > T$.

This next lemma is an analogue of Lemma 3.2 found in [11].

Lemma 4. *Let $\xi \in Lip(\frac{1}{2})$ with $\|\xi\|_{\frac{1}{2}} < 4$ and $\xi(0) = 0$. There exists some constant $A_0 > 0$, depending only on $\|\xi\|_{\frac{1}{2}}$, so that if $0 \leq x < y < z$ with $y - x = z - y$, then*

$$\frac{1}{A_0} \leq \frac{\phi(x) - \phi(y)}{\phi(y) - \phi(z)} \leq A_0. \tag{7}$$

To prove this lemma, we will need the following.

Lemma 5. *Let $c < 4$ and $0 < \epsilon < 1$. Then there exists $\delta > 0$ so that*

$$\frac{\phi(\beta)}{\phi(\alpha)} \geq 1 + \delta$$

for non-zero α and β satisfying $\frac{\beta}{\alpha} \geq 1 + \epsilon$ and for any $Lip(\frac{1}{2})$ driving term ξ with $\|\xi\|_{\frac{1}{2}} \leq c$.

Proof. Notice first that without loss of generality we can take $\alpha = -1$ and $\beta \leq -(1 + \epsilon)$ by the scaling property.

Suppose there is no such δ as in the statement of the lemma. Then for each $n \in \mathbb{N}$ there exists a driving term ξ_n and $\beta_n \leq -(1+\epsilon)$ so that $b_n < (1 + \frac{1}{n})a_n$, where $0 < a_n := \phi(-1) < b_n := \phi(\beta_n)$. Set $T_n = T(a_n)$ and $S_n = T(b_n)$.

By Ascoli-Arzela, there exists a subsequence of $\{\xi_n\}$ which converges locally uniformly to ξ. Note that $\xi \in \mathrm{Lip}(\frac{1}{2})$ with $\|\xi\|_{\frac{1}{2}} \leq c$. Since $T(x) \asymp x^2$ by Corollary 1, a_n, b_n, β_n, T_n, and S_n are all bounded. Hence by taking subsequences and renaming to avoid notational hazards, we have $a_n \to a, b_n \to b, \beta_n \to \beta, T_n \to T$, and $S_n \to S$. Note that $a = b$ since $a_n < b_n < (1 + \frac{1}{n})a_n$. If we had that $T(a) =$

$T = T(-1)$ and $T(b) = S = T(\beta)$, this would give us the desired contradiction, since $T(-1) < T(\beta)$. The same argument can be used to prove each of these four equalities, and so we will simply show that $T(a) = T$. Since $\xi_n \to \xi$ locally uniformly, $\xi_n(T_n) \to \xi(T)$. Hence $\lim_{n\to\infty} a_n(T_n) = \lim_{n\to\infty} \xi_n(T_n) = \xi(T)$, where $a_n(t)$ is the solution to (6) with $a_n(0) = a_n$. Thus it remains to show that $a_n(T_n) \to a(T)$.

Claim: Let $\epsilon > 0$. Then $a_n(T-\epsilon) \to a(T-\epsilon)$.

Proof of Claim: We will assume without loss of generality that $T_n \geq T - \frac{\epsilon}{2}$. Then, $a_n(T-\epsilon)$ is well-defined and is bounded away from $\xi_n(T-\epsilon)$ by a factor of $\sqrt{\epsilon}$ by Corollary 1.

Fix n for a moment. Then looking to solve the initial value problem (6) with the method of successive approximations, let $\psi_0^n \equiv a_n$ and recursively define

$$\psi_{k+1}^n(t) = a_n + \int_0^t \frac{-2}{\psi_k^n(s) - \xi_n(s)} ds.$$

Similarly, let ψ_k be the approximation for ξ with initial value a. Then for $t \in [0, T-\epsilon]$, $\psi_k^n(t) \geq a_n(t)$ and $\psi_k(t) \geq a(t)$. By an easy induction argument, we have that for $t \in [0, T-\epsilon]$,

$$|\psi_k^n(t) - \psi_k(t)| \leq |a_n - a| + (|a_n - a| + \|\xi_n - \xi\|_\infty) \sum_{j=1}^{k} \frac{(Bt)^j}{j!}$$

where B depends only on ϵ. So, for $t \in [0, T-\epsilon]$,

$$\begin{aligned}|a_n(t) - a(t)| &= \lim_{k\to\infty} |\psi_k^n(t) - \psi_k(t)| \\ &\leq |a_n - a| + (|a_n - a| + \|\xi_n - \xi\|_\infty)(\mathrm{e}^{Bt} - 1).\end{aligned}$$

Therefore, $a_n(T-\epsilon) \to a(T-\epsilon)$, proving the claim.

Assuming $T_n \in [T - \frac{\epsilon}{2}, T + \frac{\epsilon}{2}]$ and using Corollary 1, we have

$$\begin{aligned}0 \leq a_n(T-\epsilon) - a_n(T_n)& \\ &= (a_n(T-\epsilon) - \xi_n(T-\epsilon)) + (\xi_n(T-\epsilon) - \xi_n(T_n)) \\ &\leq A\sqrt{\epsilon} + c\sqrt{T_n - (T-\epsilon)} \\ &\leq A\sqrt{\epsilon}\end{aligned}$$

where A is a constant depending only on c. So by the claim above,

$$0 \leq a(T-\epsilon) - \lim_{n\to\infty} a_n(T_n) \leq A\sqrt{\epsilon}$$

implying that $a_n(T_n) \to a(T)$.

□

Now we are ready for the proof of Lemma 4.

Proof. In this proof, $A \geq 1$ will stand for any constant which depends only on $\|\xi\|_{\frac{1}{2}}$. Let $z(t)$ be the solution to (6) with $z(0) = z$, and $\hat{z}(t)$ the solution to (6) with $\hat{z}(0) = \phi(z)$. Define $x(t)$, $y(t)$, $\hat{x}(t)$ and $\hat{y}(t)$ similarly.

First we consider the case $x = 0$. Instead of only taking $z = 2y$, we simply assume that $\frac{z}{y} \in [1+\epsilon, 2]$, since we will reduce the next case to this setting. By the scaling invariance, we can assume that $y = 1$. Set $T = T(1)$, and recall that $K_1 \leq T \leq K_2$ from Corollary 1. Then $z(T) - \xi(T) \leq 2 + c\sqrt{K}_2$. Abusing notation a little, we have $T(z) = T + T(z(T) - \xi(T))$, where by $T(z(T) - \xi(T))$ we mean the hitting time for the solution to (6) with initial value $z(T)$ and driving term $\xi(T+t)$. By Corollary 1,

$$\phi(z)^2 \leq \frac{1}{K_1} T(\phi(z)) = \frac{1}{K_1} T(z) \leq \frac{K_2}{K_1}(1 + (2 + c\sqrt{K}_2)^2)$$

and similarly,

$$\phi(1)^2 \geq \frac{1}{K_2} T(\phi(1)) = \frac{1}{K_2} T(1) \geq \frac{K_1}{K_2}.$$

Therefore,

$$\frac{\phi(z)}{\phi(1)} \leq A.$$

By Lemma 5, we have

$$\frac{\phi(z)}{\phi(1)} \geq 1 + \delta$$

where δ depends only on c and ϵ. This gives (7) in the case $x = 0$.

Next we consider the case where $x > 0$ and $z \geq 2x$. We will reduce this to case 1 by letting time run for $T = T(x)$ at which point $x(T) = \xi(T)$. Since

$$\frac{\partial}{\partial t} \log\left(\frac{y(t) - x(t)}{z(t) - y(t)}\right) = 2\frac{z(t) - x(t)}{(x(t) - \xi(t))(y(t) - \xi(t))(z(t) - \xi(t))},$$

the quotient $q(t) := \frac{y(t)-x(t)}{z(t)-y(t)}$ is increasing in t. Therefore $q(T) > 1$. Also,

$$q(T) = \frac{y(T) - x(T)}{z(T) - y(T)} \leq \frac{y + c\sqrt{T}}{\frac{1}{2}(z - x)} \leq \frac{(1 + c\sqrt{K}_2)z}{\frac{1}{4}z} \leq A.$$

Now we are back to case 1, since we have $(1 + \frac{1}{A})(y(T) - \xi(T)) \leq z(T) - \xi(T) \leq 2(y(T) - \xi(T))$. Hence by case 1, there exists A depending only on c, so that

$$\frac{1}{A} \leq \frac{\hat{x}(T) - \hat{y}(T)}{\hat{y}(T) - \hat{z}(T)} \leq A.$$

Now we would like to run time from T back to 0 to give (7) for case 2. Since the quotient will be decreasing in t as time run backward, we immediately get the upper bound. For the lower bound,

$$\frac{\phi(x) - \phi(y)}{\phi(y) - \phi(z)} \geq \frac{\phi(x) - \phi(y)}{\hat{y}(T) - \hat{z}(T)} \geq \frac{1}{A}\frac{\phi(x) - \phi(y)}{\hat{x}(T) - \hat{y}(T)} \geq \frac{1}{A}\frac{\phi(x) - \phi(y)}{-\phi(y)} \geq \frac{1}{A},$$

where Lemma 5 gives the last inequality. Therefore (7) holds for case 2.

While these first two cases required more work than in the situation in [11], the final case where $x > 0$ and $z - x < x$ follows the arguments of Lemma 3.2 in [11] without any complications. The idea, similar to the strategy used in the previous case, is to let time run for S, where S is the first time that $x(S) - \xi(S) = z(S) - x(S)$, and to show that the quotient $q(t)$ is bounded on $[0, S]$. Thus, we end up in a setting similar to case 2. It remains then to verify that case 2 still applies and to run time backwards from S to 0, again utilizing the boundedness of $q(t)$.

□

We include the statement of Lemma 2.2 from [11] below, since we will use it in the proof of Theorem 3. It gives a condition in terms of the welding homeomorphism for when a slit-halfplane is a quasislit-halfplane.

Lemma 6. *$\mathbb{H} \setminus \gamma[0,T]$ is a quasislit-halfplane if and only if there is a constant $1 \leq M < \infty$ such that*

$$\frac{1}{M} \leq \frac{x - \xi(0)}{\xi(0) - \phi(x)} \leq M$$

for all $x > \xi(0)$ and

$$\frac{1}{M} \leq \frac{\phi(x) - \phi(y)}{\phi(y) - \phi(z)} \leq M$$

whenever $\xi(0) \leq x < y < z$ with $y - x = z - y$. Furthermore, the quasislit constant K of $\mathbb{H} \setminus \gamma[0,T]$ depends on M only.

Proof of Theorem 3

Proof. By the scaling property, it suffices to show that $f_1(\mathbb{H})$ is a quasislit plane. Let $n \in \mathbb{N}$, and set $t_k = k/n$. Following the methods in [11], we wish to construct $\xi_n \in \mathrm{Lip}(\frac{1}{2})$ so that $\xi_n(t_k) = \xi(t_k)$ and $\|\xi_n\|_{\frac{1}{2}} \leq c := \|\xi\|_{\frac{1}{2}}$. There are at least two ways to proceed. The first is by linear interpolation, and this is the method we will use. Alternatively, setting $c_k = (\xi(t_k) - \xi(t_{k+1}))\sqrt{n}$, we can define $\hat{\xi}_n(t)$ for $t \in [0,1]$ by $\hat{\xi}_n|_{[t_k,t_{k+1}]}(t) = c_k\sqrt{t_{k+1} - t} + \xi(t_{k+1})$. Although $\hat{\xi}_n \in \mathrm{Lip}(\frac{1}{2})$, it may not be true that $\|\hat{\xi}_n\|_{\frac{1}{2}} \leq c$. However, it is possible to complete the proof using this construction for $\hat{\xi}_n$ by considering the larger space of locally $\mathrm{Lip}(\frac{1}{2})$ functions and verifying that all the lemmas remain true for these functions as well. The benefit to using this construction is that we know slightly more about the generated domains. If $\hat{\phi}^k_t$ is the map generated by $\hat{\xi}_n(t_k + t) = c_k\sqrt{\frac{1}{n} - t} + \alpha_{k+1}$ for $t \in [0, \frac{1}{n}]$, then $\hat{\phi}^k_{\frac{1}{n}}$ is a map from $\mathbb{H}$ onto the upper halfplane slit by a line segment whose angle with the real line is bounded away from 0 and π.

Using our first method of linear interpolation, we set $m_k = n(\xi(t_{k+1}) - \xi(t_k))$ and define $\xi_n(t)$ for $t \in [0,1]$ by $\xi_n|_{[t_k,t_{k+1}]}(t) = m_k(t - t_k) + \xi(t_k)$. First we check that $\|\xi_n\|_{\frac{1}{2}} \leq c$. Let $x, y \in [0,1]$. If $x, y \in [t_k, t_k + 1]$ for some k, then clearly

$|\xi_n(y) - \xi_n(x)| \leq c\sqrt{|y-x|}$. So assume that $t_j \leq x \leq t_{j+1} \leq t_k \leq y \leq t_{k+1}$, and assume without loss of generality that $\xi_n(y) \geq \xi_n(x)$. If we maximize the function $h(x,y) := \xi_n(y) - \xi_n(x) - c\sqrt{y-x}$ over $(x,y) \in [t_j, t_{j+1}] \times [t_k, t_{k+1}]$, we find that $h(x,y) \leq 0$, as desired.

Let ϕ_t^k be the maps generated by $\xi_n(t_k + t) = m_k t + \xi(t_k)$ for $t \in [0, \frac{1}{n}]$. Then $\phi^k := \phi^k_{\frac{1}{n}}$ is a map from $\mathbb{H}$ onto the upper halfplane slit by a smooth curve which makes an angle of $\frac{\pi}{2}$ with the real line. If f_t^n is the map generated by ξ_n for $t \in [0,1]$, we have that $f_1^n = \phi^n \circ \phi^{n-1} \circ \cdots \circ \phi^2 \circ \phi^1$. Hence, $f_1^n(\mathbb{H})$ is a slit-halfplane. By Corollary 1, the first condition of Lemma 6 is satisfied, while the second condition is a result of Lemma 4. Therefore, we have that $f_1^n(\mathbb{H})$ is a K-quasislit-halfplane, with K independent of n. By compactness of the space of K-quasislit-halfplanes, we have that $f_1(\mathbb{H})$ is a quasislit-halfplane.

$\square$

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